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Sum of Hermitian Matrices with Given Eigenvalues: Inertia, Rank, and Multiple Eigenvalues

Chi-Kwong Li and Yiu-Tung Poon

Abstract. Let A and B be $n \times n$ complex Hermitian (or real symmetric) matrices with eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$. All possible inertia values, ranks, and multiple eigenvalues of $A + B$ are determined. Extension of the results to the sum of k matrices with $k > 2$ and connections of the results to other subjects such as algebraic combinatorics are also discussed.

1 Introduction

Let \mathcal{H}_n be the real linear space of $n \times n$ complex Hermitian (or real symmetric) matrices. For a real vector $\mathbf{a} = (a_1, \dots, a_n)$ with $a_1 \geq \cdots \geq a_n$, let

$$\mathcal{H}_n(\mathbf{a}) = \{A \in \mathcal{H}_n : A \text{ has eigenvalues } a_1, \dots, a_n\}.$$

Motivated by problems in pure and applied subjects, there has been much research on the relation between the eigenvalues of $A, B \in \mathcal{H}_n$ and those of $A + B$ [3–5, 7–9, 11, 12]. In particular, for given real vectors $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, and $\mathbf{c} = (c_1, \dots, c_n)$ with entries arranged in descending order, a necessary and sufficient condition for the existence of $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $A + B \in \mathcal{H}_n(\mathbf{c})$, or equivalently,

$$(1.1) \quad \mathcal{H}_n(\mathbf{c}) \subseteq \mathcal{H}_n(\mathbf{a}) + \mathcal{H}_n(\mathbf{b}),$$

can be completely described in terms of the equality

$$(1.2) \quad \sum_{j=1}^n (a_j + b_j - c_j) = 0$$

and a collection of inequalities in the form

$$(1.3) \quad \sum_{r \in R} a_r + \sum_{s \in S} b_s \geq \sum_{t \in T} c_t$$

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for certain m element subsets $R, S, T \subseteq \{1, \dots, n\}$ with $1 \leq m < n$ determined by the Littlewood–Richardson rules; see [5, 7] for details. Using (1.2) and (1.3), we can also deduce the inequalities

$$(1.4) \quad \sum_{r \in R^c} a_r + \sum_{s \in S^c} b_s \leq \sum_{t \in T^c} c_t,$$

where R^c denotes the complement of R in $\{1, 2, \dots, n\}$. The study has connections to many different areas such as representation theory, algebraic geometry, and algebraic combinatorics.

The set of inequalities in (1.3) grows exponentially with n . Therefore, in spite of the existence of a complete description of the eigenvalues of $A + B$ in terms of those of A and B in \mathcal{H}_n , it is sometimes difficult to answer some basic questions related to the eigenvalues of the matrices A , B , and $A + B$. For example, as pointed out by Fulton [7, p.215], given a *proper* subset K of $\{1, 2, \dots, n\}$ and real numbers $\{\gamma_k : k \in K\}$, it is not easy to use the inequalities in (1.3) to determine if there exists \mathbf{c} with $c_k = \gamma_k$ for all $k \in K$ such that (1.1) holds. In particular, the inequalities in (1.3) with $T \subseteq K$ together with those in (1.4) with $T^c \subseteq K$ are necessary, but not sufficient, for (1.1) in general.

If $K = \{k\}$ is a singleton, then inequalities in (1.3) and (1.4) reduce to Weyl's inequalities [13], implying that $c_k \in [L_k, R_k]$, where

$$L_k = \max\{a_i + b_j : i + j = n + k\} \quad \text{and} \quad R_k = \min\{a_i + b_j : i + j = k + 1\}.$$

Conversely, one can check that for every $c \in [L_k, R_k]$, there exists

$$(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$$

satisfying $A + B \in \mathcal{H}_n(\mathbf{c})$ with $c_k = c$. So, in this case, the inequalities in (1.3) with $T \subseteq K$ and $c_k = \gamma_k$ for $k \in K$ are also sufficient.

In this paper, we show that if $\mu \in [L_k, L_{k-1}] \cap (R_{k'+1}, R_{k'}]$, then there exists $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $C = A + B$ has a vector of eigenvalues \mathbf{c} with

$$c_{k-1} < \mu = c_k = c_{k+1} = \dots = c_{k'} < c_{k'+1}.$$

This will follow from a consequence (Corollary 5.7) of the solution of the following problem.

Problem 1 Suppose $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. Can a given $\mu \in \mathbf{R}$ be an eigenvalue of $A + B$ with a specific multiplicity? Equivalently, can $A + B - \mu I$ have a specific rank?

We will study the following more difficult problem.

Problem 2 Suppose $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. Can a given $\mu \in \mathbf{R}$ be an eigenvalue of $A + B$ so that p other eigenvalues are larger than μ , and q other eigenvalues are smaller than μ ? Equivalently, can $A + B - \mu I$ have inertia $(p, q, n - p - q)$, i.e., p positive eigenvalues, q negative eigenvalues, and $n - p - q$ zero eigenvalues?

Clearly, one can replace (A, B) by $(A - \mu I, B)$ and replace $\mathbf{a} = (a_1, \dots, a_n)$ by $(a_1 - \mu, \dots, a_n - \mu)$ so as to focus on the case for $\mu = 0$ in the study.

For two nonnegative integers p and q with $p + q \leq n$, let

$$\mathcal{H}_n(p, q) = \{X \in \mathcal{H}_n : X \text{ has } p \text{ positive eigenvalues and } q \text{ negative eigenvalues}\}.$$

In Section 2, we determine a necessary and sufficient condition on (p, q) for the existence of $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ so that $A + B \in \mathcal{H}_n(p, q)$. In addition, we give a global description of the set of integer pairs (p, q) satisfying these conditions in Section 3. Moreover, we determine those integer pairs (p, q) for the existence of diagonal matrices $A \in \mathcal{H}_n(\mathbf{a})$ and $B \in \mathcal{H}_n(\mathbf{b})$ such that $A + B \in \mathcal{H}_n(p, q)$ in Section 4. Then the results are used to determine all the possible ranks of matrices of the form $A + B$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ in Section 5. We also determine the function $f: \mathbf{R} \rightarrow \mathbf{Z}$ such that $f(\mu)$ is the minimum rank of a matrix of the form $A + B - \mu I$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. Additional remarks and problems are mentioned in Section 6.

Alternatively, one can describe the results as follows. For $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$, we determine the condition on (p, q) such that $U^*AU + V^*BV \in \mathcal{H}_n(p, q)$ for some unitary matrices U and V , and use the result to determine all possible ranks and multiplicities of eigenvalues of all matrices of the form $U^*AU + V^*BV$.

It turns out that it is more convenient to state and prove the results for $A - B$. We will do this in our discussion and focus on the set

$$\text{In}(\mathbf{a}, \mathbf{b}) = \{(p, q) \in \mathbf{Z} \times \mathbf{Z} : \exists (A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b}) \text{ such that } A - B \in \mathcal{H}_n(p, q)\}.$$

We always assume that $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, and $\mathbf{c} = (c_1, \dots, c_n)$ are real vectors with entries arranged in descending order unless specified otherwise.

2 Characterization of Elements in $\text{In}(\mathbf{a}, \mathbf{b})$

First, we obtain an easy to check necessary and sufficient condition for $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$.

Theorem 2.1 *Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be real vectors with entries arranged in descending order. Suppose p and q are nonnegative integers satisfying $p + q \leq n$. Then $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ if and only if*

- (i) $(a_1, \dots, a_{n-q}) - (b_{q+1}, \dots, b_n)$ is a nonnegative vector with at least p positive entries, and
- (ii) $(b_1, \dots, b_{n-p}) - (a_{p+1}, \dots, a_n)$ is a nonnegative vector with at least q positive entries.

Moreover, if (i) and (ii) hold, then there exist block diagonal matrices

$$A = A_1 \oplus \dots \oplus A_{p+q} \in \mathcal{H}_n(\mathbf{a}) \quad \text{and} \quad B = B_1 \oplus \dots \oplus B_{p+q} \in \mathcal{H}_n(\mathbf{b})$$

with the same block sizes such that $A_j - B_j$ is rank one positive definite for $j = 1, \dots, p$ and $A_j - B_j$ is rank one negative semi-definite for $j = p + 1, \dots, p + q$.

Remark 2.2 For fixed $p, q \geq 0$ with $p + q \leq n$, let $K = \{p + 1, \dots, n - q\}$. The necessity of condition (i) and (ii) in Theorem 2.1 can be deduced from the inequalities in (1.3) with $T \subseteq K$ and $c_k = 0$ for $k \in K$. We will give a direct proof of this result for completeness.

It is convenient to use the following notation in our discussion. Suppose $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{v} = (v_1, \dots, v_m)$ are real vectors with entries arranged in descending order. Write $\mathbf{u} \geq_k \mathbf{v}$ (respectively, $\mathbf{u} >_k \mathbf{v}$) if $\mathbf{u} - \mathbf{v}$ is a nonnegative vector with at least (respectively, exactly) k positive entries. We will use $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} > \mathbf{v}$ for $\mathbf{u} \geq_0 \mathbf{v}$ and $\mathbf{u} >_0 \mathbf{v}$, respectively. For $\mathbf{a} = (a_1, \dots, a_n)$ and $1 \leq m \leq n$, let $\mathbf{a}^m = (a_1, \dots, a_m)$ and $\mathbf{a}_m = (a_{n-m+1}, \dots, a_n)$. One can use these notations to restate conditions (i) and (ii) in Theorem 2.1 as

$$\mathbf{a}^{n-q} \geq_p \mathbf{b}_{n-q} \quad \text{and} \quad \mathbf{b}^{n-p} \geq_q \mathbf{a}_{n-p}.$$

The following lemmas are needed to prove Theorem 2.1. The first one was proved in [6].

Lemma 2.3 Let $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_m)$ and $\mathbf{a} = (a_1, \dots, a_n)$ be real vectors with entries arranged in descending order, where $1 \leq m < n$. Then there is $(A, \tilde{A}) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_m(\tilde{\mathbf{a}})$ with \tilde{A} as the leading principal submatrix of A if and only if $a_j \geq \tilde{a}_j \geq a_{n-m+j}$ for $j = 1, \dots, m$.

Lemma 2.4 Let $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. If $A - B$ is a rank k positive semi-definite matrix, then $\mathbf{a} \geq_k \mathbf{b}$.

Proof Applying a suitable unitary similarity to $A - B$, we may assume that $A - B = \text{diag}(d_1, \dots, d_k, 0, \dots, 0)$ with $d_1 \geq \dots \geq d_k > 0$. Let $C = B + d_k I_k \oplus 0_{n-k}$ have eigenvalues $c_1 \geq \dots \geq c_n$. Then using the positive semi-definite ordering, we have

$$A \geq C \quad \text{and} \quad B + d_k I \geq C \geq B.$$

By Weyl's inequalities (see [13]), we have

$$a_j \geq c_j \quad \text{and} \quad b_j + d_k \geq c_j \geq b_j, \quad j = 1, \dots, n.$$

Since

$$kd_k = \text{tr}(C - B) = \sum_{j=1}^n (c_j - b_j),$$

and each of the summands on the right side is bounded by d_k , we see that at least k of the summands are positive. It follows that there are at least k indices j such that $a_j > b_j$. ■

Lemma 2.5 Let \mathbf{a} and \mathbf{b} be real vectors. Suppose $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ can be partitioned as

$$\{a_1, a_2, \dots, a_n\} = \bigcup_{j=1}^r \{a_{j,1}, \dots, a_{j,n_j}\} \quad \text{and} \quad \{b_1, b_2, \dots, b_n\} = \bigcup_{j=1}^r \{b_{j,1}, \dots, b_{j,n_j}\}$$

such that for each $1 \leq j \leq r$,

$$a_{j,1} \geq b_{j,1} \geq a_{j,2} \geq b_{j,2} \geq \cdots \geq a_{j,n_j} \geq b_{j,n_j}$$

with $a_{j,i} > b_{j,i}$ for at least k_j i 's and $\sum_{j=1}^r k_j \geq m$. Then there exist block diagonal matrices $A = A_1 \oplus \cdots \oplus A_m \in \mathcal{H}_n(\mathbf{a})$ and $B = B_1 \oplus \cdots \oplus B_m \in \mathcal{H}_n(\mathbf{b})$ with the same block sizes such that $A_j - B_j$ is rank one positive definite for $j = 1, \dots, m$. Consequently, $(m, 0) \in \text{In}(\mathbf{a}, \mathbf{b})$.

Proof Suppose $r = 1$. We prove the statement by induction on m . When $m = 1$ we have

$$(2.5) \quad a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_n \geq b_n$$

and $a_i > b_i$ for at least one i . If $b_n \geq 0$, then by Lemma 2.4 there is $\tilde{A} \in \mathcal{H}_{n+1}$ with eigenvalues $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ such that the leading $n \times n$ submatrix has eigenvalues $b_1 \geq \cdots \geq b_n$. Since \tilde{A} is singular, there is $R \in M_n$ and $v \in \mathbb{C}^n$ such that $\tilde{A} = [R|v]^* [R|v]$. Note that $B = RR^*$ and R^*R have the same eigenvalues $b_1 \geq \cdots \geq b_n$, and the eigenvalues of $A = [R|v][R|v]^* = RR^* + vv^*$ are the same as the n largest of \tilde{A} and equal to $a_1 \geq \cdots \geq a_n$. Thus, there exists unitary $A - B = vv^*$ is rank one positive semi-definite. If $b_n < 0$, apply the argument to $A - b_n I$ and $B - b_n I$ to get the conclusion.

Suppose the result holds for some $m \geq 1$, and (2.5) holds with $a_i > b_i$ for at least $m + 1$ i 's. Let $s = \min\{i : a_i > b_i\}$. Then by induction assumption, there exist $A_1, B_1 \in \mathcal{H}_s$ with eigenvalues a_1, \dots, a_s and b_1, \dots, b_s and block diagonal matrices $A_2 \oplus \cdots \oplus A_{m+1}$ and $B_2 \oplus \cdots \oplus B_{m+1} \in \mathcal{H}_{n-s}$ with eigenvalues a_{s+1}, \dots, a_n and b_{s+1}, \dots, b_n such that $A_j - B_j$ is rank one positive definite for $j = 1, \dots, m + 1$. Thus, $A = A_1 \oplus A_2 \oplus \cdots \oplus A_{m+1} \in \mathcal{H}_n(\mathbf{a})$ and $B = B_1 \oplus B_2 \oplus \cdots \oplus B_{m+1} \in \mathcal{H}_n(\mathbf{b})$ satisfy the requirement.

Now, suppose $r > 1$. Choose non-negative numbers ℓ_j with $\min\{1, k_j\} \leq \ell_j \leq k_j$ for $1 \leq j \leq m$ such that $\ell_1 + \cdots + \ell_m = m$. By the result when $r = 1$, there exist block diagonal matrices A_j and $B_j \in \mathcal{H}_{n_j}$ with eigenvalues $a_{j,1}, \dots, a_{j,n_j}$ and $b_{j,1}, \dots, b_{j,n_j}$ such that $A_j - B_j$ is positive semi-definite with rank ℓ_j . Thus, for $A = A_1 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus \cdots \oplus B_m$, $A - B$ is positive semi-definite with rank m . ■

We are now ready to present the proof.

Proof of Theorem 2.1 Suppose $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ satisfies $A - B \in \mathcal{H}_n(p, q)$. Applying a unitary similarity to $A - B$, we may assume that $A - B = \text{diag}(c_1, \dots, c_n)$ such that $c_1 \geq \cdots \geq c_p > 0 = c_{p+1} = \cdots = c_{n-q} = 0 > c_{n-q+1} \geq \cdots \geq c_n$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with $A_{11}, B_{11} \in \mathcal{H}_{n-q}$. Then $A_{11} - B_{11}$ is positive semi-definite with p positive eigenvalues. Suppose A_{11} and B_{11} have eigenvalues $\alpha_1 \geq \cdots \geq \alpha_{n-q}$ and $\beta_1 \geq \cdots \geq \beta_{n-q}$, respectively. By Lemmas 2.3 and 2.4, we have

$$(a_1, \dots, a_{n-q}) \geq (\alpha_1, \dots, \alpha_{n-q}) \geq_p (\beta_1, \dots, \beta_{n-q}) \geq (b_{q+1}, \dots, b_n).$$

This proves (i). Similarly, we can prove condition (ii).

To prove the converse, given real vectors \mathbf{a} and \mathbf{b} , we first show that for every n , the result holds if $pq = 0$ or $p + q = n$. If $(p, q) = (0, 0)$, then we have $\mathbf{a} = \mathbf{b}$ and the result follows.

Suppose $p > 0$ and $q = 0$. Let $n = rp + s$, with $r \geq 0$ and $1 \leq s \leq p$ (not $0 \leq s < p$ as given by the Euclidean algorithm). Then (i) and (ii) imply that

$$\begin{aligned} a_i &\geq b_i \geq a_{p+i} \geq \cdots \geq a_{rp+i} \geq b_{rp+i} && \text{for } 1 \leq i \leq s, \\ a_j &\geq b_j \geq a_{p+j} \geq \cdots \geq a_{(r-1)p+j} \geq b_{(r-1)p+j} && \text{for } s+1 \leq j \leq p, \end{aligned}$$

with $a_i > b_i$ for at least p i 's. Therefore the result follows from Lemma 2.5.

Similarly, the result holds for $p = 0$ and $q > 0$. Hence, the result holds if $pq = 0$.

For $p + q = n$, let $A = \text{diag}(a_1, \dots, a_n)$ and $B = \text{diag}(b_{q+1}, \dots, b_n, b_1, \dots, b_{n-p})$. Then it follows from (i) and (ii) that $A - B \in \mathcal{H}_n(p, q)$.

We complete the proof of the converse by induction on n . The result is clear for $n \leq 2$.

Assume that the result is valid for all real vectors of lengths less than n . Suppose $(p, q) \geq (1, 1)$, $p + q < n$, and the inequalities in (i) and (ii) hold. Then we have

$$(2.6) \quad a_i \geq b_{q+i} \quad \text{for } 1 \leq i \leq n - q$$

and

$$(2.7) \quad b_i \geq a_{p+i} \quad \text{for } 1 \leq i \leq n - p$$

with at least p strict inequalities holding in (2.6) and at least q strict inequalities holding in (2.7).

If $a_i = b_{q+i}$ for some $1 \leq i \leq n - q$, then letting $\mathbf{a}' = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ and $\mathbf{b}' = (b_1, \dots, b_{q+i-1}, b_{q+i+1}, \dots, b_n)$, we have

$$(2.8) \quad \begin{aligned} 1 \leq j < i &\Rightarrow a'_j = a_j \geq b_{q+j} = b'_{q+j}, \\ i \leq j \leq n - 1 - q &\Rightarrow a'_j = a_{j+1} \geq b_{q+j+1} = b'_{q+j}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} 1 \leq j < i - p &\Rightarrow b'_j = b_j \geq a_{p+j} = a'_{p+j}, \\ i - p \leq j < i + q &\Rightarrow b'_j = b_j \geq a_{p+j} \geq a_{p+j+1} = a'_{p+j}, \\ i + q \leq j \leq n - 1 - p &\Rightarrow b'_j = b_{j+1} \geq a_{p+j+1} = a'_{p+j}, \end{aligned}$$

with at least p strict inequalities holding in (2.8) and at least q strict inequalities holding in (2.9). By the induction hypothesis, there exist $A', B' \in \mathcal{H}_{n-1}$ with eigenvalues $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$ and $b_1, \dots, b_{q+i-1}, b_{q+i+1}, \dots, b_n$ such that $A' - B' \in \mathcal{H}_{n-1}(p, q)$. Hence, $[a_i] \oplus A' - [b_{q+i}] \oplus B' \in \mathcal{H}_n(p, q)$.

Similarly, the result holds if $b_i = a_{p+i}$ for some $1 \leq i \leq n - p$.

So, we may assume that all inequalities are strict in (2.6) and (2.7). By symmetry, we may assume that $q \leq p$. Since $n > p + q$, let $n = r(p + q) + s$, where $r > 0$ and

$1 \leq s \leq p+q$. We will arrange a_1, \dots, a_n and b_1, \dots, b_n in $p+q$ chains of inequalities so that Lemma 2.5 can be applied. To this end, define $m = \min\{s, q, p+q-s\}$,

$$i_1 = \max\{1, s-q+1\}, \quad i_2 = \min\{s, p\}, \quad j_1 = \max\{1, s-p+1\}, \quad j_2 = \min\{s, q\}.$$

We have

	$1 \leq s \leq q$	$q < s \leq p$	$p < s \leq p+q$
$i_1 = \max\{1, s-q+1\}$	1	$s-q+1$	$s-q+1$
$i_2 = \min\{s, p\}$	s	s	p
$j_1 = \max\{1, s-p+1\}$	1	1	$s-p+1$
$j_2 = \min\{s, q\}$	s	q	q
$m = \min\{s, q, p+q-s\}$	s	q	$p+q-s$

Then $i_2 - i_1 = j_2 - j_1 = m - 1$. By conditions (i) and (ii), we can list all the entries of **a** and **b** in the following $p+q$ chains of interlacing inequalities:

$$\begin{array}{ccccccc}
 a_1 & > & b_{q+1} & > & a_{p+q+1} & > & \cdots > & b_{(r-1)(p+q)+q+1} & > & a_{r(p+q)+1} & > & b_{r(p+q)+q+1} \\
 \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 a_{i_1-1} & > & b_{q+i_1-1} & > & a_{p+q+i_1-1} & > & \cdots > & b_{(r-1)(p+q)+q+i_1-1} & > & a_{r(p+q)+i_1-1} & > & b_{r(p+q)+q+i_1-1} \\
 a_{i_1} & > & b_{q+i_1} & > & a_{p+q+i_1} & > & \cdots > & b_{(r-1)(p+q)+q+i_1} & > & a_{r(p+q)+i_1} & & \\
 \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \\
 a_{i_2} & > & b_{q+i_2} & > & a_{p+q+i_2} & > & \cdots > & b_{(r-1)(p+q)+q+i_2} & > & a_{r(p+q)+i_2} & & \\
 a_{i_2+1} & > & b_{q+i_2+1} & > & a_{p+q+i_2+1} & > & \cdots > & b_{(r-1)(p+q)+q+i_2+1} & & & & \\
 \vdots & & \vdots & & \vdots & & & & \vdots & & & & \\
 a_p & > & b_{q+p} & > & a_{p+q+p} & > & \cdots > & b_{r(p+q)}, & & & &
 \end{array}$$

and

$$\begin{array}{ccccccc}
 b_1 & > & a_{p+1} & > & b_{p+q+1} & > & \cdots > & a_{(r-1)(p+q)+p+1} & > & b_{r(p+q)+1} & > & a_{r(p+q)+p+1} \\
 \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 b_{j_1-1} & > & a_{p+j_1-1} & > & b_{p+q+j_1-1} & > & \cdots > & a_{(r-1)(p+q)+p+j_1-1} & > & b_{r(p+q)+j_1-1} & > & a_{r(p+q)+p+j_1-1} \\
 b_{j_1} & > & a_{p+j_1} & > & b_{p+q+j_1} & > & \cdots > & a_{(r-1)(p+q)+p+j_1} & > & b_{r(p+q)+j_1} & & \\
 \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots & & \\
 b_{j_2} & > & a_{p+j_2} & > & b_{p+q+j_2} & > & \cdots > & a_{(r-1)(p+q)+p+j_2} & > & b_{r(p+q)+j_2} & & \\
 b_{j_2+1} & > & a_{p+j_2+1} & > & b_{p+q+j_2+1} & > & \cdots > & a_{(r-1)(p+q)+p+j_2+1} & & & & \\
 \vdots & & \vdots & & \vdots & & & & \vdots & & & & \\
 b_q & > & a_{p+q} & > & b_{p+q+q} & > & \cdots > & a_{r(p+q)}, & & & &
 \end{array}$$

where a_i and b_i would not appear if $i < 0$ or $i > n$.

In fact, it is easy to construct the p chains of inequalities in the first list and q chains of inequalities in the second list as follows. Put the first p entries of **a** vertically in the first column of the first list, the next q entries of **a** vertically in the second column of the second list, then the next p entries of **a** in the third column of first list, and so

forth. Similarly, put the first q entries of \mathbf{b} in the first column of the second list, the next p entries of \mathbf{b} in the second column of the first list, then the next q entries of \mathbf{b} in the third column of the second list, and so forth.

For the application of Lemma 2.5, the chains of inequalities with starting terms a_i for $i_1 \leq i \leq i_2$ are not acceptable because the first and last terms are entries of \mathbf{a} . Similarly, the chains of inequalities with starting terms b_j for $j_1 \leq j \leq j_2$ are not acceptable. Since $i_2 - i_1 = j_2 - j_1$, we can amend the situations as follows. For $i_1 \leq i \leq i_2$, let $i' = j_1 + i - i_1$. Then $j_1 \leq i' \leq j_2$ and we can replace the pair of interlacing inequalities

$$\begin{aligned} a_i &> b_{q+i} > a_{p+q+i} > \cdots > b_{(r-1)(p+q)+q+i} > a_{r(p+q)+i}, \\ b_{i'} &> a_{p+i'} > b_{p+q+i'} > \cdots > a_{(r-1)(p+q)+p+i'} > b_{r(p+q)+i'}, \end{aligned}$$

by one of the following pairs:

$$\begin{aligned} a_i &> b_{q+i} > a_{p+q+i} > \cdots > b_{(r-1)(p+q)+q+i} > a_{r(p+q)+i} > b_{r(p+q)+i'}, \\ b_{i'} &> a_{p+i'} > b_{p+q+i'} > \cdots > a_{(r-1)(p+q)+p+i'} \end{aligned}$$

if $a_{r(p+q)+i} > b_{r(p+q)+i'}$, or

$$\begin{aligned} a_i &> b_{q+i} > a_{p+q+i} > \cdots > b_{(r-1)(p+q)+q+i}, \\ b_{i'} &> a_{p+i'} > b_{p+q+i'} > \cdots > a_{(r-1)(p+q)+p+i'} > b_{r(p+q)+i'} \geq a_{r(p+q)+i} \end{aligned}$$

if $a_{r(p+q)+i} \leq b_{r(p+q)+i'}$. After the above modification, we can apply Lemma 2.5 to the eigenvalues in the interlacing inequalities with starting terms a_i to get a rank p positive semi-definite matrix, and then apply Lemma 2.5 to the eigenvalues in the interlacing inequalities with starting terms b_j to get a rank q semi-definite matrix. The result follows.

Following our proof, one can construct the matrices A and B in block diagonal forms as asserted in the last statement of the theorem. ■

It is easy to use Theorem 2.1 to test whether a given pair of integers (p, q) belongs to $\text{In}(\mathbf{a}, \mathbf{b})$. Here is an example.

Example 2.6 Let $\mathbf{a} = (6, 6, 4, 3, 3, 2, 1)$ and $\mathbf{b} = (5, 4, 3, 3, 1, 1, 1)$. Then the following hold.

(i) $(1, 1) \notin \text{In}(\mathbf{a}, \mathbf{b})$ as

$$(b_1, \dots, b_{7-1}) - (a_{1+1}, \dots, a_7) = (5, 4, 3, 3, 1, 1) - (6, 4, 3, 3, 2, 1)$$

has a negative entry.

(ii) $(2, 0) \in \text{In}(\mathbf{a}, \mathbf{b})$ as

$$\begin{aligned} (a_1, \dots, a_{7-0}) - (b_{1+0}, \dots, b_7) &= (6, 6, 4, 3, 3, 2, 1) - (5, 4, 3, 3, 1, 1, 1) \\ &= (1, 2, 1, 0, 2, 1, 0) \end{aligned}$$

$$(b_1, \dots, b_{7-2}) - (a_{2+1}, \dots, a_7) = (5, 4, 3, 3, 1) - (4, 3, 3, 2, 1) = (1, 1, 0, 1, 0).$$

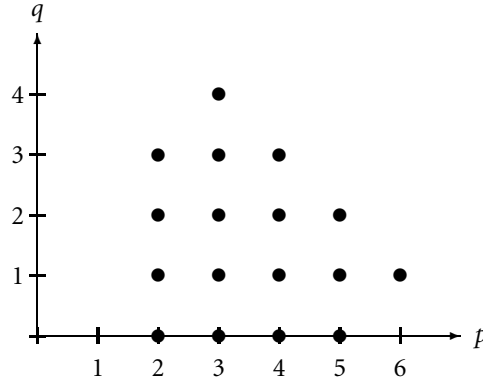
In fact, if $A = \text{diag}(6, 4, 6, 2, 3, 3, 1)$ and $B = B_1 \oplus B_2$ with

$$B_1 = \begin{pmatrix} 7/2 & \sqrt{15}/2 \\ \sqrt{15}/2 & 5/2 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 7/2 & \sqrt{5}/2 \\ \sqrt{5}/2 & 3/2 \end{pmatrix} \oplus \text{diag}(3, 3, 1),$$

then $(A, B) \in \mathcal{H}_7(\mathbf{a}) \times \mathcal{H}_7(\mathbf{b})$ such that

$$A - B = \begin{pmatrix} 5/2 & -\sqrt{15}/2 \\ -\sqrt{15}/2 & 3/2 \end{pmatrix} \oplus \left[\begin{pmatrix} 5/2 & -\sqrt{5}/2 \\ -\sqrt{5}/2 & 1/2 \end{pmatrix} \oplus \text{diag}(0, 0, 0) \right] \in \mathcal{H}_7(2, 0).$$

We can also test every (p, q) pair of nonnegative integers with $p + q \leq 7$ and depict the set $\text{In}(\mathbf{a}, \mathbf{b})$ as points in \mathbf{R}^2 as follows.



Corollary 2.7 Suppose $(p_1, q_1), (p_2, q_2) \in \text{In}(\mathbf{a}, \mathbf{b})$. Let $p = \min\{p_1, p_2\}$ and $q = \min\{q_1, q_2\}$. Then $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$.

Proof Suppose $p = p_i$ and $q = q_j$. Since $(p_1, q_1), (p_2, q_2) \in \text{In}(\mathbf{a}, \mathbf{b})$, we have

$$\mathbf{a}^{n-q_j} \geq_{p_j} \mathbf{b}_{n-q_j} \Rightarrow \mathbf{a}^{n-q} \geq_p \mathbf{b}_{n-q},$$

$$\mathbf{b}^{n-p_i} \geq_{q_i} \mathbf{a}_{n-p_i} \Rightarrow \mathbf{b}^{n-p} \geq_q \mathbf{a}_{n-p}.$$

Hence, by Theorem 2.1, $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$. ■

3 A Global Description of $\text{In}(\mathbf{a}, \mathbf{b})$

While Theorem 2.1 allows us to test if a pair of nonnegative integers lies in $\text{In}(\mathbf{a}, \mathbf{b})$, it would be nice to have a global description of the region for all integer pairs in $\text{In}(\mathbf{a}, \mathbf{b})$. The objective of this section is to obtain such a description.

Note that if \mathbf{a} and \mathbf{b} have a common entry with multiplicities n_1 and n_2 in the two vectors such that $n_1 + n_2 > n$, then for any $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$, the null space of $A - B$ has dimension at least $n_1 + n_2 - n$, and a reduction of the vectors \mathbf{a} and \mathbf{b} is possible in the problem of describing $\text{In}(\mathbf{a}, \mathbf{b})$ as shown in the following proposition.

Proposition 3.1 Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be two real vectors with entries arranged in descending order. Suppose

$$a_i = a_{i+1} = \dots = a_{i+n_1-1} = b_j = b_{j+1} = \dots = b_{j+n_2-1},$$

for some $i, j, n_1, n_2 \geq 1$ such that $n_1 + n_2 > n$. Let $s = n_1 + n_2 - n$ and \mathbf{a}', \mathbf{b}' be obtained by deleting s a_i from each of \mathbf{a} and \mathbf{b} . Then $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ if and only if $(p, q) \in \text{In}(\mathbf{a}', \mathbf{b}')$.

Proof Suppose A and B have eigenvalues a_1, \dots, a_n and b_1, \dots, b_n . Then the intersection of the eigenspaces of A and B associated with a_i has dimension $\geq s$. So there exists a unitary U such that $U^*AU = A' \oplus a_i I_s$ and $U^*BU = B' \oplus a_i I_s$. Therefore, $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ if and only if $(p, q) \in \text{In}(\mathbf{a}', \mathbf{b}')$. ■

By the above lemma, to describe $\text{In}(\mathbf{a}, \mathbf{b})$, we can focus on the (\mathbf{a}, \mathbf{b}) pair such that \mathbf{a} and \mathbf{b} do not have a common entry whose multiplicities in the two vectors have sum exceeding n . To describe the main result in this section, we need the following definition.

Definition 3.2 Suppose $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are real vectors with entries arranged in descending order. Let

$$(3.1) \quad p_0 = \begin{cases} n & \text{if } b_1 < a_n, \\ \min\{t : 0 \leq t < n, \mathbf{b}^{n-t} \geq \mathbf{a}_{n-t}\} & \text{otherwise;} \end{cases}$$

$$(3.2) \quad q_0 = \begin{cases} n & \text{if } a_1 < b_n, \\ \min\{t : 0 \leq t < n, \mathbf{a}^{n-t} \geq \mathbf{b}_{n-t}\} & \text{otherwise.} \end{cases}$$

Suppose

$$(3.3) \quad (a_1, \dots, a_n, b_1, \dots, b_n) \text{ has no entry with multiplicity larger than } n.$$

Let

$$k = \begin{cases} n - p_0 & \text{if } b_1 \leq a_n, \\ \min\{t : 0 \leq t < n - p_0, \mathbf{b}^{n-p_0-t} > \mathbf{a}_{n-p_0-t}\} & \text{otherwise;} \end{cases}$$

$$\ell = \begin{cases} n - q_0 & \text{if } a_1 \leq b_n, \\ \min\{t : 0 \leq t < n - q_0, \mathbf{a}^{n-q_0-t} > \mathbf{b}_{n-q_0-t}\} & \text{otherwise.} \end{cases}$$

Furthermore, for $0 \leq i \leq n - (p_0 + q_0 + \ell)$ and $0 \leq j \leq n - (p_0 + q_0 + k)$, let

Q_i be the number of positive entries in $\mathbf{b}^{n-p_i} - \mathbf{a}_{n-p_i}$ with $p_i = p_0 + i$,

P_j be the number of positive entries in $\mathbf{a}^{n-q_j} - \mathbf{b}_{n-q_j}$ with $q_j = q_0 + j$.

In Example 2.6, we have $(k, \ell) = (1, 1)$,

$$\begin{aligned} (p_0, q_0) &= (2, 0), & (p_0, Q_0) &= (2, 3), & (P_0, q_0) &= (5, 0), \\ (p_1, Q_1) &= (3, 4) = (P_4, q_4), & (p_2, Q_2) &= (4, 3) = (P_3, q_3), \\ (p_3, Q_3) &= (5, 2) = (P_2, q_2), & (p_4, Q_4) &= (6, 1) = (P_1, q_1). \end{aligned}$$

In general, we will show in Lemma 3.11 that $p_k \leq P_\ell$ and $p_i + Q_i = n = P_j + q_j$ for all $k \leq i \leq n - (p_0 + q_0 + \ell)$ and $\ell \leq j \leq n - (p_0 + q_0 + k)$. Therefore, the points in

$$\{(p_i, Q_i) : k \leq i \leq n - (p_0 + q_0 + \ell)\} \cup \{(P_j, q_j) : \ell \leq j \leq n - (p_0 + q_0 + k)\}$$

lie on the line segment joining (p_k, Q_k) and (P_ℓ, q_ℓ) .

Theorem 3.3 *Let \mathbf{a} and \mathbf{b} be real vectors satisfying condition (3.3). Use the notation in Definition 3.2. The following conditions hold.*

(i) *The polygon \mathcal{P} obtained by joining the points*

$$(p_0, q_0), (p_0, Q_0), (p_1, Q_1), \dots, (p_k, Q_k), (P_\ell, q_\ell), (P_{\ell-1}, q_{\ell-1}), \dots, (P_0, q_0), (p_0, q_0)$$

is convex.

(ii) *$\text{In}(\mathbf{a}, \mathbf{b})$ consists of all the integer pairs (p, q) in \mathcal{P} .*

In Example 2.6, \mathcal{P} is obtained by joining $(2, 0), (2, 3), (3, 4), (6, 1), (5, 0), (2, 0)$. Before presenting the proof of the theorem, we illustrate how to use the theorem in the following corollaries.

Corollary 3.4 *Let \mathbf{a} and \mathbf{b} be real vectors with no common entries. Using the notation in (3.1) and (3.2), we have*

$$\text{In}(\mathbf{a}, \mathbf{b}) = \{(p, q) : p \geq p_0, q \geq q_0, p + q \leq n\}.$$

Proof Since \mathbf{a} and \mathbf{b} have no common entries, we see that for each $i \in \{1, \dots, k\}$, the vector $\mathbf{b}^{n-p_i} - \mathbf{a}_{n-p_i}$ is positive, and hence $p_i + Q_i = n$. Similarly, $P_j + q_j = n$ for each $j \in \{1, \dots, \ell\}$. By Theorem 3.3, the result follows. ■

Corollary 3.5 *Suppose there are $\mu > \nu$ and $0 \leq u, v \leq n$ such that*

$$\mu = a_1 = \dots = a_u = b_1 = \dots = b_v \text{ and } \nu = a_{u+1} = \dots = a_n = b_{v+1} = \dots = b_n.$$

Then

$$\text{In}(\mathbf{a}, \mathbf{b}) = \{(u - w, v - w) : \max\{0, u + v - n\} \leq w \leq \min\{u, v\}\}.$$

Proof Without loss of generality, we may assume that $u \geq v$, $\mu = 1$ and $\nu = 0$. Furthermore, by Proposition 3.1, we may assume that $u + v = n$. Then $(p_0, q_0) = (u - v, 0)$. Moreover, $(p_i, Q_i) = (p_0 + i, i) = (P_i, q_i)$ for $i = 1, \dots, v$. By Theorem 3.3, the result follows. ■

We establish some lemmas to prove Theorem 3.3. The first three lemmas give additional properties of p_0, q_0, P_i, Q_j , and confirm that $(p_0, q_0), (p_i, Q_i), (P_j, q_j) \in \text{In}(\mathbf{a}, \mathbf{b})$.

Lemma 3.6 Suppose \mathbf{a}, \mathbf{b} are two real vectors, and p_0, q_0 are defined by (3.1) and (3.2). Then the following conditions hold.

- (i) $p_0 = \min\{p : (p, q) \in \text{In}(\mathbf{a}, \mathbf{b}) \text{ for some } q \geq 0\}$, and $\mathbf{a}^{p_0} - \mathbf{b}_{p_0}$ is a positive vector if $p_0 > 0$.
- (ii) $q_0 = \min\{q : (p, q) \in \text{In}(\mathbf{a}, \mathbf{b}) \text{ for some } p \geq 0\}$, and $\mathbf{b}^{q_0} - \mathbf{a}_{q_0}$ is a positive vector if $q_0 > 0$.
- (iii) $(p_0, q_0) \in \text{In}(\mathbf{a}, \mathbf{b})$.

Proof (i) Suppose p_0 is given by (3.1). If $p_0 = n$, then $b_1 < a_n$ and $\text{In}(\mathbf{a}, \mathbf{b}) = \{(n, 0)\}$. If $p_0 < n$, then we have $b_j \geq a_{p_0+j}$ for all $1 \leq j \leq n - p_0$. Let $A = \text{diag}(a_1, \dots, a_n)$ and $B = \text{diag}(b_{n-p_0+1}, \dots, b_n, b_1, \dots, b_{n-p_0})$. Then $A - B$ has at most p_0 positive eigenvalues. Therefore,

$$p_0 \geq \min\{p : (p, q) \in \text{In}(\mathbf{a}, \mathbf{b}) \text{ for some } q \geq 0\}.$$

On the other hand, suppose $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ for some $q \geq 0$. Then there exists $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $A - B \in \mathcal{H}_n(p, q)$. By Theorem 2.1, we have $\mathbf{b}^{n-p} \geq \mathbf{a}_{n-p}$. Therefore, $p \geq p_0$. Hence,

$$p_0 \leq \min\{p : (p, q) \in \text{In}(\mathbf{a}, \mathbf{b}) \text{ for some } q \geq 0\}.$$

If $p_0 > 0$, then there exists $1 \leq i \leq n - (p_0 - 1)$ such that $a_{p_0-1+i} > b_i$. So for all $1 \leq j \leq p_0$ we have $a_j \geq a_{p_0-1+i} > b_i \geq b_{n-p_0+j}$, i.e., $\mathbf{a}^{p_0} - \mathbf{b}_{p_0}$ is positive. This proves (i). The proof of (ii) is similar.

(iii) By the results in (i) and (ii), we can choose $p \geq p_0$ and $q \geq q_0$ such that (p, q_0) and $(p_0, q) \in \text{In}(\mathbf{a}, \mathbf{b})$. Hence, by Corollary 2.7, $(p_0, q_0) \in \text{In}(\mathbf{a}, \mathbf{b})$. ■

Note that assumption (3.3) is not needed in Lemma 3.6.

Lemma 3.7 Suppose \mathbf{a} and \mathbf{b} are real vectors satisfying condition (3.3). Let $s \in \{0, \dots, n-1\}$ be such that $\mathbf{b}^{n-s} - \mathbf{a}_{n-s}$ has a non-positive entry. Then $\mathbf{a}^{s+1} - \mathbf{b}_{s+1}$ is positive.

Proof Suppose the conclusion is not true. Then $\mathbf{a}^{s+1} - \mathbf{b}_{s+1}$ is not positive. Hence there is $i \in \{1, \dots, s+1\}$ such that $a_i \leq b_{n-s-1+i}$. Since the vector $\mathbf{b}^{n-s} - \mathbf{a}_{n-s}$ has a non-positive entry, $b_j \leq a_{s+j}$ for some $j \in \{1, \dots, n-s\}$. Hence

$$b_j = a_{s+j} \leq a_{s+j-1} \leq \dots \leq a_i \leq b_{n-s-1+i} \leq \dots \leq b_j.$$

Consequently, all the inequalities become equalities, and the multiplicity of $a_i = b_j$ in the vector $(a_1, \dots, a_n, b_1, \dots, b_n)$ equals $(s+j-i+1) + (n-s+i-j) = n+1$, contradicting assumption (3.3). ■

By Lemma 3.7 and the definition of ℓ and k , we see that

$$(n - q_0 - \ell, q_0 + \ell), (n - p_0 - k, p_0 + k) \in \text{In}(\mathbf{a}, \mathbf{b})$$

if \mathbf{a} and \mathbf{b} satisfy (3.3).

Lemma 3.8 *Let \mathbf{a} and \mathbf{b} be real vectors satisfying (3.3). Use the notation in Definition 3.2. For $0 \leq i \leq n - (p_0 + q_0 + \ell)$ and $0 \leq j \leq n - (p_0 + q_0 + k)$, we have*

- (i) $\mathbf{a}^{p_0+i} > \mathbf{b}_{p_0+i}$ and $\mathbf{b}^{q_0+j} > \mathbf{a}_{q_0+j}$.
- (ii) $(p_i, Q_i), (P_j, q_j) \in \text{In}(\mathbf{a}, \mathbf{b})$.
- (iii) $Q_i = \max\{q : (p_0 + i, q) \in \text{In}(\mathbf{a}, \mathbf{b})\}$ and $P_j = \max\{p : (p, q_0 + j) \in \text{In}(\mathbf{a}, \mathbf{b})\}$.
- (iv) $p_0 + q_0 + k + \ell \leq n$.

Proof If p_0 or $q_0 = n$, then $k = \ell = 0$, and the results follow. Therefore, in the rest of the proof, we assume that $p_0, q_0 < n$.

(i) It follows from the definitions of ℓ and k that $\mathbf{a}^{n-q_0-\ell} > \mathbf{b}_{n-q_0-\ell}$ and $\mathbf{b}^{n-p_0-k} > \mathbf{a}_{n-p_0-k}$. For $0 \leq i \leq n - (p_0 + q_0 + \ell)$ we have $p_0 + i \leq n - q_0 - \ell$. Therefore, $\mathbf{a}^{p_0+i} > \mathbf{b}_{p_0+i}$. Similarly, $\mathbf{b}^{q_0+j} > \mathbf{a}_{q_0+j}$ for $0 \leq n - (p_0 + q_0 + k)$.

(ii) Since, $\text{diag}(a_1, \dots, a_n) - \text{diag}(b_{n-p_i+1}, \dots, b_n, b_1, \dots, b_{n-p_i}) \in \mathcal{H}_n(p_i, Q_i)$, we have $(p_i, Q_i) \in \text{In}(\mathbf{a}, \mathbf{b})$. Similarly, $(P_j, q_j) \in \text{In}(\mathbf{a}, \mathbf{b})$.

(iii) Suppose $(p_i, q) \in \text{In}(\mathbf{a}, \mathbf{b})$. Then $\mathbf{b}^{n-p_i} \geq_q \mathbf{a}_{n-p_i}$. So, $q \leq Q_i$. Hence,

$$Q_i = \max\{q : (p_0 + i, q) \in \text{In}(\mathbf{a}, \mathbf{b})\}.$$

Similarly, we have

$$P_j = \max\{p : (p, q_0 + j) \in \text{In}(\mathbf{a}, \mathbf{b})\}.$$

(iv) Since $(n - q_0 - \ell, q_0 + \ell) \in \text{In}(\mathbf{a}, \mathbf{b})$, we have $n - q_0 - \ell \geq p_0$ by Lemma 3.6. From the definition of k and $\mathbf{a}^{n-q_0-\ell} > \mathbf{b}_{n-q_0-\ell}$, we have $n - q_0 - \ell \geq p_0 + k$. Thus, $p_0 + q_0 + k + \ell \leq n$. \blacksquare

Clearly, P_j is equal to $n - q_j$ minus the number of zero entries in $\mathbf{a}^{n-q_j} - \mathbf{b}_{n-q_j}$. Therefore, in order to study the relationship between P_j and P_{j+1} , we need to keep track of the zero entries in the vector $\mathbf{a}^{n-q_j} - \mathbf{b}_{n-q_j}$ and investigate how they are related to the entries of $\mathbf{a}^{n-q_{j-1}} - \mathbf{b}_{n-q_{j-1}}$. For this reason, we introduce the following definition.

Definition 3.9 For $1 \leq i \leq j \leq m \leq n$, we say that $[i, j] = \{t : i \leq t \leq j\}$ is a *maximal interval* of $(\mathbf{a}^m, \mathbf{b}_m)$ if

$$a_{i-1} > a_i = a_{i+1} = \dots = a_j = b_{n-m+i} = b_{n-m+i+1} = \dots = b_{n-m+j} > b_{n-m+j+1}.$$

The length of a maximal interval $[i, j]$ is given by $j - i + 1$. The set of all maximal intervals of $(\mathbf{a}^m, \mathbf{b}_m)$ will be denoted by $S(\mathbf{a}^m, \mathbf{b}_m)$. Let $T = T(\mathbf{a}^m, \mathbf{b}_m)$ be the maximum length of a maximal interval of $(\mathbf{a}^m, \mathbf{b}_m)$. For $1 \leq t \leq T$, let s_t be the number of maximal intervals of $(\mathbf{a}^m, \mathbf{b}_m)$ with length t . The sequence (s_1, s_2, \dots, s_T) will be denoted by $\mathbf{s}(\mathbf{a}^m, \mathbf{b}_m)$.

Lemma 3.10 Suppose $\mathbf{a}^m \geq \mathbf{b}_m$ for some $1 \leq m \leq n$. Then the following conditions hold.

- (i) $\mathbf{a}^m >_q \mathbf{b}_m$ where $q = m - \sum_{t=1}^T t s_t$.
- (ii) $[i, j] \in S(\mathbf{a}^{m-1}, \mathbf{b}_{m-1})$ if and only if $[i, j+1] \in S(\mathbf{a}^m, \mathbf{b}_m)$.
- (iii) $\mathbf{a}^{m-1} >_{q_1} \mathbf{b}_{m-1}$, where $q_1 = q - 1 + \sum_{t=1}^T s_t$.
- (iv) If $\mathbf{a}^{m-2} >_{q_2} \mathbf{b}_{m-2}$, then $q_2 - q_1 \leq q_1 - q$.

Here, we assume that $m > 1$ for (ii)–(iii) and $m > 2$ for (iv).

Proof Condition (i) holds because $\sum_{t=1}^T t s_t$ is the number of zero entries in $\mathbf{a}^m - \mathbf{b}_m$.

To prove (ii), suppose $[i, j] \in S(\mathbf{a}^{m-1}, \mathbf{b}_{m-1})$. Then we have

$$(3.4) \quad \begin{aligned} a_{i-1} &> a_i = a_{i+1} = \cdots = a_j \\ &= b_{n-(m-1)+i} = b_{n-(m-1)+i+1} = \cdots = b_{n-(m-1)+j} > b_{n-(m-1)+j+1}. \end{aligned}$$

Since

$$\begin{aligned} a_{i-1} &> a_i \geq b_{n-m+i} \geq b_{n-m+i+1} = a_i, \\ a_j &\geq a_{j+1} \geq b_{n-m+j+1} = a_j > b_{n-(m-1)+j+1}, \end{aligned}$$

we have $a_i = b_{n-m+i} = b_{n-m+i+1}$ and $a_j = a_{j+1} = b_{n-m+j+1}$. This gives

$$(3.5) \quad \begin{aligned} a_{i-1} &> a_i = a_{i+1} = \cdots = a_{j+1} \\ &= b_{n-m+i} = b_{n-m+i+1} = \cdots = b_{n-m+j+1} > b_{n-m+j+2}. \end{aligned}$$

Thus, $[i, j+1] \in S(\mathbf{a}^m, \mathbf{b}_m)$. Conversely, if $[i, j+1] \in S(\mathbf{a}^m, \mathbf{b}_m)$ for some $j \geq i$, then (3.5) holds. Thus (3.4) follows and $[i, j] \in S(\mathbf{a}^{m-1}, \mathbf{b}_{m-1})$.

To prove (iii), let $\mathbf{s}(\mathbf{a}^m, \mathbf{b}_m) = (s_1, s_2, \dots, s_T)$. Then it follows from (ii) that $\mathbf{s}(\mathbf{a}^{m-1}, \mathbf{b}_{m-1}) = (s_2, s_3, \dots, s_T)$. Hence,

$$q_1 = m - 1 - \sum_{t=2}^T (t-1) s_t = m - 1 - \sum_{t=1}^T t s_t + \sum_{t=1}^T s_t = q - 1 + \sum_{t=1}^T s_t.$$

From (iii) we have $q_2 - q_1 = \sum_{t=2}^T s_t - 1 \leq \sum_{t=1}^T s_t - 1 = q_1 - q$. This proves (iv). ■

Applying Lemma 3.10 to the quantities in Definition 3.2, we readily deduce the following.

Lemma 3.11 Using the notation in Definitions 3.2 and 3.9, the following conditions hold.

- (i) $k = T(\mathbf{b}_{n-p_0}, \mathbf{a}_{n-p_0})$, $\ell = T(\mathbf{a}_{n-q_0}, \mathbf{b}_{n-q_0})$.

- (ii) Suppose $\mathbf{s}(\mathbf{b}_{n-p_0}, \mathbf{a}_{n-p_0}) = (s_1, s_2, \dots, s_k)$ and $\mathbf{s}(\mathbf{a}_{n-q_0}, \mathbf{b}_{n-q_0}) = (s'_1, s'_2, \dots, s'_\ell)$. Then

$$Q_{i+1} = Q_i - 1 + \sum_{t=i+1}^k s_t \quad \text{for } 0 \leq i < k,$$

$$P_{j+1} = P_j - 1 + \sum_{t=j+1}^k s'_t \quad \text{for } 0 \leq j < \ell.$$

- (iii) For $k \leq i < n - (p_0 + q_0 + \ell)$ and $\ell \leq j < n - (p_0 + q_0 + k)$, we have

$$Q_{i+1} = Q_i - 1 \quad \text{and} \quad P_{j+1} = P_j - 1.$$

Moreover, for $k \leq i \leq n - (p_0 + q_0 + \ell)$ and $\ell \leq j \leq n - (p_0 + q_0 + k)$, we have

$$p_i + Q_i = n = P_j + q_j.$$

- (iv) For $0 < i < n - (p_0 + q_0 + \ell)$ and $\ell < j < n - (p_0 + q_0 + k)$, we have

$$Q_i - Q_{i-1} \geq Q_{i+1} - Q_i \quad \text{and} \quad P_j - P_{j-1} \geq P_{j+1} - P_j$$

Proof of Theorem 3.3 (i) From (p_0, q_0) to (p_0, Q_0) , we have a vertical straight line segment. Note that the slope of the line segment from (p_{i-1}, Q_{i-1}) to (p_i, Q_i) equals $Q_i - Q_{i-1}$, and the slope of the line segment from (p_i, Q_i) to (p_{i+1}, Q_{i+1}) is $Q_{i+1} - Q_i$. By Lemma 3.11(iv), we see that $Q_i - Q_{i-1} \geq Q_{i+1} - Q_i$. Thus, the polygonal curve joining the points $(p_0, Q_0), (p_1, Q_1), \dots, (p_k, Q_k)$ is convex. The line segment joining (p_k, Q_k) and (p_ℓ, q_ℓ) is a line segment with negative slope. Finally, the polygonal curve joining the points $(p_0, q_0), (p_0, Q_0), \dots, (p_\ell, q_\ell)$ is concave by Lemma 3.11(iv). Thus \mathcal{P} is a convex subset contained in the set

$$\{(p, q) : p_0 \leq p \leq n - q_\ell, q_0 \leq q \leq n - p_k, \text{ and } p + q \leq n\}.$$

- (ii) Suppose $(p, q) \in \mathcal{P}$. Let $p = p_i$ and $q = q_j$ for some $0 \leq i \leq n - (p_0 + q_0 + \ell)$ and $0 \leq j \leq n - (p_0 + q_0 + k)$. Then $p_i \leq P_j$ and $q_j \leq Q_i$. Since (p_i, Q_i) and $(P_j, q_j) \in \text{In}(\mathbf{a}, \mathbf{b})$. By Corollary 2.7, $(p_i, q_j) \in \text{In}(\mathbf{a}, \mathbf{b})$.

Conversely, suppose $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$. By Theorem 3.6, we have $p \geq p_0, q \geq q_0$ and $p + q \leq n$. Let $p = p_i$ and $q = q_j$ for some $i, j \geq 0$. If $i > n - (p_0 + q_0 + \ell)$, then we have

$$q_j \leq n - p_i < q_0 + \ell \quad \Rightarrow \quad p_i \leq P_j \leq n - q_\ell \quad \Rightarrow \quad i \leq n - (p_0 + q_0 + \ell),$$

which is a contradiction. Therefore, $0 \leq i \leq n - (p_0 + q_0 + \ell)$. Similarly, we have $0 \leq j \leq n - (p_0 + q_0 + k)$. Since $(p_i, q_j) \in \text{In}(\mathbf{a}, \mathbf{b})$, we have $p_i \leq P_j$ and $q_j \leq Q_i$ by Lemma 3.8. If either $p_i = P_j$ or $q_j = Q_i$, then $(p, q) \in \mathcal{P}$. So we may assume that $p_i < P_j$ and $q_j < Q_i$. Consider the positive numbers

$$t_1 = j(P_j - p_i), \quad t_2 = i(Q_i - q_j), \quad \text{and} \quad t_3 = (P_j - p_i)(Q_i - q_j).$$

Then by direct computation we have

$$\begin{aligned} \frac{t_1(p_i, Q_i) + t_2(P_j, q_j) + t_3(p_0, q_0)}{t_1 + t_2 + t_3} &= \frac{(t_1 p_i + t_2 P_j + t_3 p_0, t_1 Q_i + t_2 q_j + t_3 q_0)}{t_1 + t_2 + t_3} \\ &= (p_i, q_j). \end{aligned}$$

Thus, (p, q) lies in \mathcal{P} . ■

4 Elements in $\text{In}(\mathbf{a}, \mathbf{b})$ Attainable by Diagonal Matrices

In this section, we determine those elements in $\text{In}(\mathbf{a}, \mathbf{b})$ that are attainable by diagonal matrices. Clearly, if A and B are diagonal matrices with eigenvalues so that the eigenvalues of A and those of B are mutually distinct, then $A - B$ is invertible. If A and B have m common eigenvalues (counting multiplicities), then $A - B$ has at most m zero eigenvalues. It turns out that this is the only additional restriction on $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ to be attainable by diagonal matrices.

Theorem 4.1 *Suppose \mathbf{a} and \mathbf{b} have m common entries counting multiplicities. Then there are diagonal matrices $A \in \mathcal{H}_n(\mathbf{a})$ and $B \in \mathcal{H}_n(\mathbf{b})$ such that $A - B \in \mathcal{H}_n(p, q)$ if and only if $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ and $p + q \geq n - m$.*

To prove Theorem 4.1 we need the following.

Lemma 4.2 *Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{R}^n$ with*

$$a_1 \geq a_2 \geq \dots \geq a_n \quad \text{and} \quad b_1 \geq b_2 \geq \dots \geq b_n.$$

Given $1 \leq j_1 \leq i_1 \leq n$, let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be obtained from \mathbf{a} and \mathbf{b} by deleting a_{i_1} and b_{j_1} from \mathbf{a} and \mathbf{b} , respectively. Suppose $\mathbf{a} >_p \mathbf{b}$ for some $0 \leq p \leq n$. We have

- (i) $\hat{\mathbf{a}} \geq \hat{\mathbf{b}}$.
- (ii) *If $1 \leq p \leq n$, then $\hat{\mathbf{a}} \geq_{p-1} \hat{\mathbf{b}}$.*
- (iii) *If $a_i = b_i$ for some $j_1 \leq i \leq i_1$, then $\hat{\mathbf{a}} \geq_p \hat{\mathbf{b}}$.*

Proof Since

$$\hat{a}_i = \begin{cases} a_i & \text{if } 1 \leq i < i_1, \\ a_{i+1} & \text{if } i_1 \leq i \leq n-1, \end{cases} \quad \hat{b}_j = \begin{cases} b_j & \text{if } 1 \leq j < j_1, \\ b_{j+1} & \text{if } j_1 \leq j \leq n-1, \end{cases}$$

we have

$$\begin{aligned} 1 \leq i < j_1 &\Rightarrow \hat{a}_i = a_i \geq b_i = \hat{b}_i \\ j_1 \leq i < i_1 &\Rightarrow \hat{a}_i = a_i \geq b_i \geq b_{i+1} = \hat{b}_i \\ i_1 \leq i < n &\Rightarrow \hat{a}_i = a_{i+1} \geq b_{i+1} = \hat{b}_i \end{aligned}$$

and (i) holds.

Note that every strict inequality $a_i > b_i$ for $1 \leq i < i_1$ (or $i_1 < i \leq n$) gives a strict inequality $\hat{a}_i > \hat{b}_i$ (or $\hat{a}_{i-1} > \hat{b}_{i-1}$). This proves (ii) and the case when $i = i_1$ or j_1 in (iii).

For (iii), we may assume that $a_{i_1} > b_{i_1}$ and $i_1 > j_1$. Note that

$$\begin{aligned}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_{j_1-1}) &= (a_1, a_2, \dots, a_{j_1-1}), \\(\hat{b}_1, \hat{b}_2, \dots, \hat{b}_{j_1-1}) &= (b_1, b_2, \dots, b_{j_1-1}), \\(\hat{a}_{j_1}, \hat{a}_{j_1+1}, \dots, \hat{a}_{i_1-1}) &= (a_{j_1}, a_{j_1+1}, \dots, a_{i_1-1}), \\(\hat{b}_{j_1}, \hat{b}_{j_1+1}, \dots, \hat{b}_{i_1-1}) &= (b_{j_1}, b_{j_1+1}, \dots, b_{i_1-1}), \\(\hat{a}_{i_1}, \hat{a}_{i_1+1}, \dots, \hat{a}_n) &= (a_{i_1}, a_{i_1+1}, \dots, a_n), \\(\hat{b}_{i_1}, \hat{b}_{i_1+1}, \dots, \hat{b}_n) &= (b_{i_1}, b_{i_1+1}, \dots, b_n).\end{aligned}$$

Apply Lemma 3.10 (iii) to $(a_{j_1}, a_{j_1+1}, \dots, a_{i_1})$ and $(b_{j_1}, b_{j_1+1}, \dots, b_{i_1})$; by the fact that at least one s_k is positive, we can conclude that the number of strict inequalities in $(\hat{a}_{j_1}, \hat{a}_{j_1+1}, \dots, \hat{a}_{i_1-1}) - (\hat{b}_{j_1}, \hat{b}_{j_1+1}, \dots, \hat{b}_{i_1-1})$ is no less than that of

$$(a_{j_1}, a_{j_1+1}, \dots, a_{i_1}) - (b_{j_1}, b_{j_1+1}, \dots, b_{i_1}).$$

Therefore, the number of entries in $\hat{\mathbf{a}} - \hat{\mathbf{b}}$ is no less than that of $\mathbf{a} - \mathbf{b}$. ■

Proof of Theorem 4.1 Suppose A and B are diagonal matrices with eigenvalues a_1, \dots, a_n and b_1, \dots, b_n such that $A - B \in \mathcal{H}_n(p, q)$. So, $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$. Also, the number of zero diagonal entries is at most m . Therefore, $m \geq n - p - q$. Hence, $p + q \geq n - m$.

We prove the converse by induction on m . Let $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ and $p + q \geq n - m$. If $p + q = n$. Then the result follows from Theorem 2.1. So the result holds for $m = 0$ and we may assume that $n > p + q$.

Let $m > 0$. Assume the result holds whenever \mathbf{a} and \mathbf{b} have $m - 1$ entries in common. Suppose \mathbf{a} and \mathbf{b} have m common entries and $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$, with $p + q \geq n - m$. By Theorem 2.1, we have $\mathbf{a}^{n-q} \geq_p \mathbf{b}_{n-q}$ and $\mathbf{b}^{n-p} \geq_q \mathbf{a}_{n-p}$. We may assume that $n > p + q \geq n - m$. We are going to show that we can delete common entries from \mathbf{a} and \mathbf{b} to obtain vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}} \in \mathbf{R}^{n-1}$ so that $\hat{\mathbf{a}}^{n-1-q} \geq_p \hat{\mathbf{b}}_{n-1-q}$ and $\hat{\mathbf{b}}^{n-1-p} \geq_q \hat{\mathbf{a}}_{n-1-p}$. Since $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ have only $m - 1$ entries in common and $p + q \geq (n - 1) - (m - 1)$, the result will follow.

Consider the following cases:

Case 1: $\mathbf{a}^{n-q} \geq_{p+1} \mathbf{b}_{n-q}$ and $\mathbf{b}^{n-p} \geq_{q+1} \mathbf{a}_{n-p}$.

Since $m > 0$, we can choose $i_1 = \min\{i : a_i = b_j \text{ for some } j\}$ and $j_1 = \min\{j : b_j = a_{i_1}\}$. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be obtained from \mathbf{a} and \mathbf{b} by deleting a_{i_1} and b_{j_1} respectively.

If $i_1 > n - q$, then $\hat{\mathbf{a}}^{n-1-q} = \mathbf{a}^{n-1-q}$. Therefore, $\hat{\mathbf{a}}^{n-1-q} \geq_p \mathbf{b}_{n-1-q}$.

If $i_1 \leq n - q$, then $b_{j_1-1} > b_{j_1} = a_{i_1} \geq b_{q+i_1}$ and we have $q + i_1 \geq j_1$. By Lemma 4.2(ii), $\hat{\mathbf{a}}^{n-1-q} \geq_p \hat{\mathbf{b}}_{n-1-q}$.

Similarly, we have $\hat{\mathbf{b}}^{n-1-p} \geq_q \hat{\mathbf{a}}_{n-1-p}$.

Case 2: $\mathbf{a}^{n-q} >_p \mathbf{b}_{n-q}$.

Since $n - q > p$, let $i_1 = \min\{t : 1 \leq t \leq n - q \text{ and } a_t = b_{q+t}\} \leq p + 1$. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be obtained from \mathbf{a} and \mathbf{b} by deleting a_{i_1} and b_{q+i_1} , respectively. Then $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ have $m - 1$ entries in common. By Lemma 4.2(iii), $\hat{\mathbf{a}}^{n-1-q} \geq_p \hat{\mathbf{b}}_{n-1-q}$. Consider the following cases.

Subcase 2a: If $\mathbf{b}^{n-p} \geq_{q+1} \mathbf{a}_{n-p}$, then it follows from Lemma 4.2(ii) that $\hat{\mathbf{b}}^{n-1-p} \geq_q \hat{\mathbf{a}}_{n-1-p}$.

Subcase 2b: If $\mathbf{b}^{n-p} >_q \mathbf{a}_{n-p}$, then

$$\min\{s : 1 \leq s \leq n - p \text{ and } b_s = a_{p+s}\} \leq q + 1 \leq q + i_1.$$

It follows from Lemma 4.2(iii) that $\hat{\mathbf{b}}^{n-1-p} \geq_q \hat{\mathbf{a}}_{n-1-p}$. ■

5 Ranks and Multiple Eigenvalues

By Theorem 3.3, we can determine the set $R(\mathbf{a}, \mathbf{b})$ of all possible ranks for a matrix of the form $A - B$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. Evidently, we have

$$R(\mathbf{a}, \mathbf{b}) = \{p + q : (p, q) \in \text{In}(\mathbf{a}, \mathbf{b})\}.$$

Nevertheless, it is interesting that the result can be put into the following simple form.

Theorem 5.1 *Let \mathbf{a}, \mathbf{b} be real vectors, and define p_0 and q_0 as in (3.1) and (3.2). Let m be the largest multiplicity of an entry in $(a_1, \dots, a_n, b_1, \dots, b_n)$ and let $r = \min\{2n - m, n\}$. Suppose $R(\mathbf{a}, \mathbf{b})$ is the set of rank values of matrices of the form $A - B$, where $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. Then one of the following holds.*

(i) *There exist real numbers $\mu > \nu$ and $u, v \in \{0, \dots, n\}$ such that*

$$\mathbf{a} = (\underbrace{\mu, \dots, \mu}_u, \nu, \dots, \nu), \quad \mathbf{b} = (\underbrace{\mu, \dots, \mu}_v, \nu, \dots, \nu),$$

and

$$R(\mathbf{a}, \mathbf{b}) = \{u + v - 2j : \max\{0, u + v - n\} \leq j \leq \min\{u, v\}\}.$$

(ii) *Condition (i) does not hold, $\mathbf{a} = \mathbf{b}$, and $R(\mathbf{a}, \mathbf{b}) = \{0\} \cup \{2, \dots, r\}$.*

(iii) *Conditions (i) and (ii) do not hold, and $R(\mathbf{a}, \mathbf{b}) = \{p_0 + q_0, \dots, r\}$.*

Moreover, if $t \in R(\mathbf{a}, \mathbf{b})$, then there are block diagonal matrices

$$A = A_1 \oplus \dots \oplus A_t \in \mathcal{H}_n(\mathbf{a}) \quad \text{and} \quad B = B_1 \oplus \dots \oplus B_t \in \mathcal{H}_n(\mathbf{b})$$

with the same block sizes such that $A_j - B_j$ has rank one for $j = 1, \dots, t$.

Note that in the theorem, we include the case when $(a_1, \dots, a_n, b_1, \dots, b_n)$ has an entry with multiplicity larger than n .

Proof (1) Suppose \mathbf{a}, \mathbf{b} satisfy the condition in (i). The result follows from Corollary 3.5.

(2) Suppose condition (i) does not hold and $\mathbf{a} = \mathbf{b}$. If $A = B = \text{diag}(a_1, \dots, a_n)$, then $A - B \in \mathcal{H}_n(0, 0)$. Since A and B have the same trace, we see that $A - B$ cannot have rank 1.

Without loss of generality, we may assume that $r = n$. We prove the following claim by induction on n :

There are matrices $A, B \in \mathcal{H}_n(\mathbf{a})$ such that $A - B \in \mathcal{H}_n(p, q)$ whenever $2 \leq p + q \leq n$ with $p = q$ or $p = q + 1$.

The claim is clear if $n = 3, 4$. Suppose $n \geq 5$ and $2 \leq p + q \leq n$ with $p = q$ or $p = q + 1$. Since \mathbf{a} has at least three distinct entries, each entry has multiplicity at most $n/2$. Suppose $a_r > a_s$, where a_r, a_s have the two largest multiplicities in the vector \mathbf{a} .

For $2 \leq p + q \leq 3$, choose $a_w \notin \{a_u, a_v\}$ and let $A_1 = \text{diag}(a_u, a_v, a_w)$. Then there exists a diagonal matrix B_1 with the same eigenvalues as A_1 and $A_1 - B_1 \in \mathcal{H}_3(p, q)$. Remove a_u, a_v, a_w from \mathbf{a} to get \mathbf{a}' . Then $A_1 \oplus \text{diag}(\mathbf{a}') - B_1 \oplus \text{diag}(\mathbf{a}') \in \mathcal{H}_n(p, q)$.

For $4 \leq p + q \leq n$, we have $p, q \geq 1$. Therefore, $2 \leq (p-1) + (q-1) \leq n-2$ and $p-1 = q-1$ or $p-1 = (q-1) + 1$. Let $A_1 = \text{diag}(a_u, a_v)$ and $B_1 = \text{diag}(a_v, a_u)$; we have $A_1 - B_1 \in \text{In}(1, 1)$. Remove a_r, a_s from \mathbf{a} to get \mathbf{a}' . Since $n \geq 5$, there are at least three distinct entries in \mathbf{a}' and each has multiplicity at most $(n-2)/2$. By the induction assumption, there are A_2, B_2 both with vector of eigenvalues \mathbf{a}' such that $A_2 - B_2 \in \mathcal{H}_{n-2}(p-1, q-1)$. Thus, $A_1 \oplus A_2 - B_1 \oplus B_2 \in \mathcal{H}_n(p, q)$.

(3) Suppose conditions (i) and (ii) do not hold. Using the notation in Theorem 3.3, we see that $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ for

$$(p, q) \in \{(p_j, q_0) : 0 \leq j \leq k\} \cup \{(p_k, q_j) : 1 \leq j \leq Q_k\}.$$

Thus, we have the desired rank values.

By Theorem 2.1, we can construct matrices A and B with the asserted block structure. ■

It is clear that $X, Y \in \mathcal{H}_n$ have the same eigenvalues if and only if $X - \mu I$ and $Y - \mu I$ have the same inertia (or rank) for all eigenvalues μ of Y . Thus, we can describe the eigenvalues of $A - B$ in terms of the inertia of $A - B - \mu I$ for different real numbers μ . In particular, we have the following necessary condition for $c_1 \geq \dots \geq c_n$ to be the eigenvalues of $A - B$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$.

Proposition 5.2 *Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, and $\mathbf{c} = (c_1, \dots, c_n)$ be real vectors with entries arranged in descending order. Suppose \mathbf{c} has distinct entries $c_1 > \dots > c_t$ with multiplicities m_1, \dots, m_t , respectively, and suppose there exists $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $A - B \in \mathcal{H}_n(\mathbf{c})$. Set $u_0 = 0$, $u_j = m_1 + \dots + m_{j-1}$ for $j \in \{1, \dots, t\}$, $v_j = m_{j+1} + \dots + m_t$ for $j \in \{1, \dots, t-1\}$ and $v_t = 0$. Then for $j \in \{1, \dots, t\}$,*

- (i) $(a_1 - c_j, \dots, a_{n-v_j} - c_j) - (b_{v_j+1}, \dots, b_n)$ is nonnegative with at least u_j positive entries.
- (ii) $(b_1, \dots, b_{n-u_j}) - (a_{u_j+1} - c_j, \dots, a_n - c_j)$ is nonnegative with at least v_j positive entries.

Remark 5.3 Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ with entries arranged in descending order. Then there exist $A, B \in \mathcal{H}_n$ with vector of eigenvalues \mathbf{a} and \mathbf{b} such that $A - B$ has an eigenvalue μ with multiplicity t if and only if there is a matrix of the form $\tilde{A} - B$ with rank $n - t$, where $\tilde{A} + \mu I \in \mathcal{H}_n(\mathbf{a})$ and $B \in \mathcal{H}_n(\mathbf{b})$. Hence, we can use Theorem 5.1 to determine whether there is $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $A - B$ has an eigenvalue μ with multiplicity t . In Corollaries 5.6 and 5.7, we will apply Theorem 2.1 to give a more precise location of the multiple eigenvalue μ . As a byproduct, we determine the function $f(\mu)$ defined as the minimum rank of a matrix of the form $A - B - \mu I$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ for given real vectors \mathbf{a} and \mathbf{b} .

The following notation will be used for the rest of this section.

Notation 5.4 Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ be real vectors with entries arranged in descending order. For $0 \leq t \leq n - 1$, let

$$\alpha_t = \max\{a_{j+t} - b_j : 1 \leq j \leq n - t\} \quad \text{and} \quad \beta_t = \min\{a_j - b_{j+t} : 1 \leq j \leq n - t\}.$$

For $\mu \in \mathbf{R}$, let $p_0(\mu)$ and $q_0(\mu)$ be defined as in (3.1), (3.2) with a_j replaced by $a_j - \mu$.

Note that $p_0(\mu) + q_0(\mu)$ will be the minimum rank of a matrix of the form $A - B - \mu I$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$.

Proposition 5.5 Let \mathbf{a} and \mathbf{b} be real vectors with entries arranged in descending order. We have

$$\alpha_{n-1} \leq \alpha_{n-2} \leq \dots \leq \alpha_0 \quad \text{and} \quad \beta_0 \leq \beta_1 \leq \dots \leq \beta_{n-1}.$$

Moreover, the following conditions hold for the functions $p_0(\mu)$, $q_0(\mu)$, and $p_0(\mu) + q_0(\mu)$.

- (i) $p_0(\mu)$ is a decreasing step function in $\mu \in \mathbf{R}$ such that $p_0(\mu) = n$ for $\mu < \alpha_{n-1}$, $p_0(\mu) = 0$ for $\mu \geq \alpha_0$, and $p_0(\mu) = t$ if μ is in the interval $[\alpha_t, \alpha_{t-1})$ for $1 \leq t \leq n - 1$;
- (ii) $q_0(\mu)$ is an increasing step function in $\mu \in \mathbf{R}$ such that $q_0(\mu) = 0$ for $\mu \leq \beta_0$, $q_0(\mu) = n$ for $\mu > \beta_{n-1}$, and $q_0(\mu) = t$ if μ is in the interval $(\beta_{t-1}, \beta_t]$ for $1 \leq t \leq n - 1$.
- (iii) If $\alpha_s = \beta_t$ for some $0 \leq s, t \leq n - 1$, then there exists $\delta > 0$ such that $p_0(\mu) + q_0(\mu) > p_0(\alpha_s) + q_0(\alpha_s)$ for all $0 < |\mu - \alpha_s| < \delta$.
- (iv) If $\mu \neq \alpha_t, \beta_t$ for all $0 \leq t \leq n - 1$, then $p_0(\cdot) + q_0(\cdot)$ is locally constant at μ .

Proof For $1 \leq t \leq n - 1$ and $1 \leq j \leq n - t$ we have $a_{j+t} - b_j \leq a_{j+(t-1)} - b_j$. Therefore, $\alpha_t \leq \alpha_{t-1}$. Similarly, $\beta_t \geq \beta_{t-1}$.

By (3.1) and (3.2), we have

$$p_0(\mu) = \begin{cases} n & \text{if } \mu < a_n - b_1, \\ \min\{t : \mu \geq \alpha_t\} & \text{otherwise,} \end{cases}$$

$$q_0(\mu) = \begin{cases} n & \text{if } a_1 - b_n < \mu, \\ \min\{t : \mu \leq \beta_t\} & \text{otherwise,} \end{cases}$$

which implies (i) and (ii).

For (iii), suppose $\alpha_s = \beta_t$ for some $0 \leq s, t \leq n-1$. By taking $\alpha_n = \alpha_{n-1} - 1$, $\alpha_{-1} = \alpha_0 + 1$, $\beta_{-1} = \beta_0 - 1$, and $\beta_n = \beta_{n-1} + 1$, we may assume that

$$\begin{aligned}\alpha_{s+1} &< \alpha_s = \alpha_{s-1} = \cdots = \alpha_{s'} < \alpha_{s'-1}, \\ \beta_{t-1} &< \beta_t = \beta_{t+1} = \cdots = \beta_{t'} < \beta_{t'+1}.\end{aligned}$$

Let $\delta = \min\{\alpha_s - \alpha_{s+1}, \alpha_{s'-1} - \alpha_{s'}, \beta_t - \beta_{t-1}, \beta_{t'+1} - \beta_{t'}\} > 0$. We have

$$\begin{aligned}p_0(\mu) + q_0(\mu) &= \begin{cases} s + t + 1 & \text{if } 0 < \alpha_s - \mu < \delta, \\ s' + t' + 1 & \text{if } 0 < \mu - \alpha_s < \delta \end{cases} \\ &> s' + t = p_0(\alpha_{s'}) + q_0(\beta_t) = p_0(\alpha_s) + q_0(\alpha_s)\end{aligned}$$

(iv) follows from (i) and (ii). ■

Note that the function $g(\mu)$ defined as the maximum rank of a matrix of the form $A - B - \mu I$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ is easy to determine, namely, it is equal to $g(\mu) = \min\{n, 2n - m(\mu)\}$ with $m(\mu)$ equal to the maximum multiplicity of an entry in the vector $(a_1 - \mu, \dots, a_n - \mu, b_1, \dots, b_n)$.

Similarly, one can consider $P_\ell(\mu)$ and $Q_k(\mu)$ defined as the maximum number of positive and negative eigenvalues of a matrix of the form $A - B - \mu I$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. We omit their discussion.

The following corollary concerns the possible multiplicities for $\mu \in \mathbf{R}$ to be an eigenvalue of $A - B$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$.

Corollary 5.6 *Let \mathbf{a} and \mathbf{b} be real vectors with entries arranged in descending order. Suppose $a_n - b_1 \leq \mu \leq a_1 - b_n$. Then there exist $s, t \in \{0, \dots, n-1\}$ such that $\mu \in [\alpha_s, \alpha_{s-1}] \cap (\beta_{t-1}, \beta_t]$, where we take $\alpha_{-1} > \beta_{n-1}$ and $\beta_{-1} < \alpha_{n-1}$.*

- (i) *Suppose $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ and μ is an eigenvalue of $A - B$. Then the multiplicity of μ is at most $n - s - t$. Furthermore, $A - B$ has at least s eigenvalue greater than μ and at least t eigenvalues less than μ .*
- (ii) *There exists $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $A - B$ has an eigenvalue μ with multiplicity $n - s - t$, s eigenvalues greater than μ , and t eigenvalues less than μ .*

To facilitate the comparison of our results and those in the literature, we present the next corollary in terms of $A + B$ with $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$. We use the following notation. Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, and $\mathbf{c} = (c_1, \dots, c_n)$ with entries arranged in descending order. For each $1 \leq k \leq n$, let

$$L_k = \max\{a_i + b_j : i + j = n + k\} \quad \text{and} \quad R_k = \min\{a_i + b_j : i + j = k + 1\}.$$

Suppose $A, B \in \mathcal{H}_n$ and $C = A + B$ have eigenvalues \mathbf{a} , \mathbf{b} , and \mathbf{c} . Then it follows from Weyl's inequalities [13] that $L_k \leq c_k \leq R_k$. Conversely, for every $1 \leq k \leq n$ and $c \in [L_k, R_k]$, there exist $A, B \in \mathcal{H}_n$, and $C = A + B$ with eigenvalues \mathbf{a} , \mathbf{b} , and \mathbf{c} such that $c_k = c$. However, for two distinct $1 \leq k < k' \leq n$ and $c \in [L_k, R_k]$, $c' \in [L_{k'}, R_{k'}]$, there may not exist $A, B \in \mathcal{H}_n$ and $C = A + B$ with eigenvalues \mathbf{a} , \mathbf{b} ,

and \mathbf{c} such that $c_k = c$ and $c_{k'} = c'$; see the example in [7, p. 215]. Nevertheless, by replacing b_j with $-b_{n+1-j}$ and putting $s = k - 1$ and $t = n - k'$, the second part of Corollary 5.6 can be rephrased in the following form.

Corollary 5.7 *Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ with entries arranged in descending order and $\mu \in [L_k, L_{k-1}) \cap (R_{k'+1}, R_{k'})$. Then there exists $(A, B) \in \mathcal{H}_n(\mathbf{a}, \mathbf{b})$ such that $C = A + B$ has a vector of eigenvalues \mathbf{c} with*

$$c_{k-1} < \mu = c_k = c_{k+1} = \dots = c_{k'} < c_{k'+1}.$$

We remark that Corollary 5.7 can also be deduced from the results in [1].

6 Additional Results and Remarks

Proposition 6.1 *Let \mathbf{a}, \mathbf{b} be given. There are $1 \times n$ vectors \mathbf{a}' and \mathbf{b}' with integral entries arranged in descending order such that $\text{In}(\mathbf{a}, \mathbf{b}) = \text{In}(\mathbf{a}', \mathbf{b}')$. Moreover, for each $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ there is $A \in \mathcal{H}_n(\mathbf{a}')$ and $B \in \mathcal{H}_n(\mathbf{b}')$ such that $A - B \in \text{In}(\mathbf{a}', \mathbf{b}')$ has integer eigenvalues.*

Proof We can construct \mathbf{a}' and \mathbf{b}' as follows. Use the entries of \mathbf{a} and \mathbf{b} to form a vector $\gamma = (\gamma_1, \dots, \gamma_{2n})$ with entries in descending order. We always put the entries of \mathbf{a} first if an entry appears in both vectors. Suppose γ has m distinct entries $\mu_1 > \dots > \mu_m$. Then replace the entries μ_i in \mathbf{a} and \mathbf{b} by the integer i for each $i \in \{1, \dots, m\}$ to get the vectors \mathbf{a}' and \mathbf{b}' . By Theorem 2.1 and the construction of \mathbf{a}' and \mathbf{b}' , we see that $(p, q) \in \text{In}(\mathbf{a}, \mathbf{b})$ if and only if $(p, q) \in \text{In}(\mathbf{a}', \mathbf{b}')$. Moreover, by Theorem 2.1, for each $(p, q) \in \text{In}(\mathbf{a}', \mathbf{b}')$ we can construct

$$A = A_1 \oplus \dots \oplus A_{p+q} \in \mathcal{H}_n(\mathbf{a}') \quad \text{and} \quad B = B_1 \oplus \dots \oplus B_{p+q} \in \mathcal{H}_n(\mathbf{b}')$$

such that $A_i - B_i$ is a rank one positive semi-definite for $i = 1, \dots, p$, and $A_i - B_i$ is a rank one negative semi-definite for $i = p+1, \dots, p+q$. Since A_i and B_i have integral eigenvalues, the only nonzero eigenvalue of $A_i - B_i$, which equals $\text{tr}(A_i - B_i)$, is again an integer. So, the last assertion holds. ■

Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}$ have nonnegative integral entries. It is known that there exist $(A, B) \in \mathcal{H}_n(\mathbf{a}) \times \mathcal{H}_n(\mathbf{b})$ such that $A - B \in \mathcal{H}_n(\mathbf{c})$ if and only if one can obtain the Young diagram associated with (a_1, \dots, a_n) from the Young diagrams associated with (b_1, \dots, b_n) and (c_1, \dots, c_n) according to the Little–Richardson rules; see [7]. Thus, we have the following result.

Proposition 6.2 *Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ have positive integral entries arranged in descending order. Then there is a vector $\mathbf{c} = (c_1, \dots, c_n)$ with positive integral entries arranged in descending order and $c_{p+1} = \dots = c_{n-q+1} = \mu$ for a given integer μ such that one can obtain the Young diagram associated with \mathbf{a} from the Young diagrams associated with \mathbf{b} and \mathbf{c} according to the Little–Richardson rules if and only if*

$$(a_1 - \mu, \dots, a_{n-q} - \mu) \geq_p (b_{q+1}, \dots, b_n)$$

and

$$(b_1, \dots, b_{n-p}) \geq_q (a_{p+1} - \mu, \dots, a_n - \mu).$$

In connection with our discussion, it would be interesting to solve the following.

Problem 3 Deduce and extend Proposition 6.2 using the theory of algebraic combinatorics. In particular, for given real vectors \mathbf{a} and \mathbf{b} with integral entries, determine the conditions for the existence of an integral vector \mathbf{c} with certain prescribed entries such that the Young diagram corresponding to \mathbf{a} can be obtained from those of \mathbf{b} and \mathbf{c} according to the Littlewood–Richardson rules.

Problem 4 Extend our results to the sum of k Hermitian matrices for $k > 2$. In other words, determine all possible inertia values and ranks of matrices in $\mathcal{H}_n(\mathbf{a}_1) + \cdots + \mathcal{H}_n(\mathbf{a}_k)$ for given $1 \times n$ real vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ with entries arranged in descending order.

Note that the problem of finding the relation between the eigenvalues of A_1, \dots, A_k and those of $A_0 = A_1 + \cdots + A_k$ can be reformulated as the problem of finding the necessary and sufficient conditions for the existence of Hermitian matrices A_0, A_1, \dots, A_k with prescribed eigenvalues such that $A_0 - \sum_{j=1}^k A_j$ has rank 0. Thus, it can be viewed as a special case of Problem 4. To determine whether there are $A_1, \dots, A_k \in \mathcal{H}_n$ with prescribed eigenvalues such that $A_1 + \cdots + A_k$ has rank one, one may reduce the problem to the study of the existence of $A_1, \dots, A_k \in \mathcal{H}_n$ with prescribed eigenvalues such that $A_1 + \cdots + A_k$ has eigenvalue $\mu, 0, \dots, 0$ with $\mu = \text{tr}(A_1 + \cdots + A_k)$. Then the results in [7] can be used to solve the problem. In general, it seems difficult to determine if there exist A_1, \dots, A_k with prescribed eigenvalues such that $A_1 + \cdots + A_k$ has rank r with $r \in \{2, \dots, n\}$.

Note added in proof We thank Professor Wing Suet Li for some helpful discussions about the connection between the interesting preprint [1] and our work. In [1, Proposition 5.1], the authors determined the conditions on $1 \times n$ vectors $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k$ with some of their entries specified so that one can fill in the missing entries to get vectors $\tilde{\mathbf{a}}_0, \dots, \tilde{\mathbf{a}}_k$ with entries arranged in descending order and Hermitian matrices $A_j \in \mathcal{H}_n(\tilde{\mathbf{a}}_j)$ for $j = 0, 1, \dots, k$ satisfying $A_0 = A_1 + \cdots + A_k$. Evidently, there exists $A_0 \in \mathcal{H}(\mathbf{a}_1) + \cdots + \mathcal{H}(\mathbf{a}_k)$ with inertia $(p, q, n - p - q)$ for given $1 \times n$ real vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ if and only if there exist $\delta, \varepsilon > 0$ and $A_0 \in \mathcal{H}(\mathbf{a}_1) + \cdots + \mathcal{H}(\mathbf{a}_k)$ with eigenvalues $\mu_1 \geq \cdots \geq \mu_n$ such that $(\mu_p, \dots, \mu_{n-q+1}) = (\delta, 0, \dots, 0, -\varepsilon)$. Using the result in [1], one can determine whether the positive numbers δ, ε exist by checking whether a polytope defined a large number of inequalities in terms of entries of $\mathbf{a}_1, \dots, \mathbf{a}_k$ has non-empty interior; see also Buch [2]. For $k = 2$, our Theorem 2.1 shows that the very involved conditions can be reduced to (i) and (ii).

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